ISRAEL JOURNAL OF MATHEMATICS 114 (1999), 359–380

A PARTITION OF CHARACTERS ASSOCIATED TO NILPOTENT SUBGROUPS*

BY

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ABSTRACT

If G is a finite solvable group and H is a maximal nilpotent subgroup of G containing $\mathbf{F}(G)$, we show that there is a canonical basis $P(G|H)$ of the space of class functions on G vanishing off any G-conjugate of H which consists of characters. Via $P(G|H)$ it is possible to partition the irreducible characters of G into "blocks". These behave like Brauer p-blocks and a Fong theory for them can be developed.

1. Introduction

Suppose that G is a finite group and let $\mathrm{cf}(G)$ be the space of complex class functions defined on G. If H is any subgroup of G, we consider the subspace

$$\mathrm{vcf}(G \mid H) = \{\chi \in \mathrm{cf}(G) \mid \chi(g) = 0 \text{ if } g \text{ does not lie in any } G\text{-conjugate of } H\}.$$

The dimension of this subspace is the number of conjugacy classes of G meeting H, and, as can be easily checked,

$$\mathrm{vcf}(G \mid H) = \{\eta^G \mid \eta \in \mathrm{cf}(H)\},$$

where η^G denotes the induced class function of η to G.

Write $\mathrm{Irr}(G)$ for the set of irreducible complex characters of G. We say that a basis $\mathcal{B}$ of $\mathrm{vcf}(G \mid H)$ is **good** if it satisfies the following two conditions:

(I) if $\eta \in \mathcal{B}$, then there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = \eta$; and

* Research partially supported by DGICYT and MEC.
Received June 17, 1998

(D) if $\gamma \in \mathrm{Irr}(H)$, then $\gamma^G = \sum_{\eta \in \mathcal{B}} a_\eta \eta$ for uniquely determined nonnegative integers a_η.

In general, good bases do not necessarily exist. If they exist, however, it is easy to see that they are necessarily unique. We will denote by $P(G \mid H)$ the unique good basis (if it exists) of $\mathrm{vcf}(G \mid H)$.

Once we have our uniquely defined basis $P(G \mid H)$ for certain H, it is natural to define linking in the set $\mathrm{Irr}(G)$ and study the associated graph (the "blocks" relative to the subgroup H). We say that $\chi, \psi \in \mathrm{Irr}(G)$ are **linked** if there exists $\eta \in P(G \mid H)$ such that

$$[\chi, \eta] \neq 0 \neq [\psi, \eta].$$

If H is a Hall p-complement of a p-solvable group G, then there exists $P(G \mid H)$ and this is the set of projective indecomposable characters by a celebrated theorem of P. Fong. Of course, linking partitions $\mathrm{Irr}(G)$ into the Brauer p-blocks. When H is a Hall π-subgroup of a π-separable group G, then $P(G \mid H)$ also exists and is the set of projective indecomposable characters associated to the Isaacs π-partial characters $\mathrm{I}_\pi(G)$. Linking, in this case, partitions $\mathrm{Irr}(G)$ into the Isaacs–Slattery π-blocks.

If G is a finite solvable group and H is a maximal nilpotent subgroup of G containing $\mathbf{F}(G)$, the Fitting subgroup of G (that is, if H is a **nilpotent injector** of G), we proved in [5] that $P(G \mid H)$ exists. If we say that $\chi, \psi \in \mathrm{Irr}(G)$ are N-**linked** if there exists $\eta \in P(G \mid H)$ such that

$$[\chi, \eta] \neq 0 \neq [\psi, \eta],$$

then the N-**blocks** of G are the connected components in $\mathrm{Irr}(G)$ of the graph defined by N-linking.

It is perhaps surprising that there exists a well behaved theory of N-blocks which resembles Fong's theory on p-blocks of p-solvable groups in which the p'-radical $\mathbf{O}_{p'}(G)$ is replaced by the Fitting subgroup $\mathbf{F}(G)$ and the p-complements of G by the nilpotent injectors of G.

The N-blocks of G are inductively described via a "Fong–Reynolds type" Theorem A and Theorem B below. (Recall that if $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G \mid \theta)$ is the set of irreducible characters of G lying over θ.)

THEOREM A: *Let G be a solvable group, let N be a normal nilpotent subgroup of G and let B be an* N*-block of G.*

(i) *There exists $\theta \in \mathrm{Irr}(N)$ such that $B \subseteq \mathrm{Irr}(G \mid \theta)$.*

(ii) *If T is the stabilizer of θ in G, then there exists an* N*-block b of T such that $b \subseteq \mathrm{Irr}(T \mid \theta)$ and*

$$B = \{\psi^G \mid \psi \in b\}.$$

THEOREM B: *Let G be a solvable group and let $\theta \in \mathrm{Irr}(\mathbf{F}(G))$ be G-invariant. Then $\mathrm{Irr}(G \mid \theta)$ is an* N*-block of G.*

Once we have defined N-blocks for every finite solvable group G, a number of questions naturally appear. We hope that some of them might suggest interesting problems.

THEOREM C: *Suppose that G is a solvable group, let B be an* N*-block of G and let H be a nilpotent injector of G. Then*

$$|B| \leq |G : H|.$$

The analogy between Hall π-subgroups and nilpotent injectors suggests that the latter might have some interesting character theory to be developed. The next result (although not difficult to prove) seems not to have been noticed up to now.

THEOREM D: *Let G be solvable. Then the set of elements of G lying in some nilpotent injector H of G and $|H|$ are determined in the character table of G.*

In order to prove the next result, however, we shall use our results on good bases.

THEOREM E: *Let H be a nilpotent injector of a finite solvable group G. Suppose that λ and μ are linear characters of H. Then $\lambda^G = \mu^G$ if and only if $\lambda = \mu^x$ for some $x \in \mathbf{N}_G(H)$.*

2. Inertia groups and injectors

In [8], we developed some theorems which are useful for finding good bases for the spaces $\mathrm{vcf}(G \mid H)$. In Section 3 below, we will apply these results to the case where G is solvable and H is a nilpotent injector of G. For proving these theorems, the following is a key definition.

Suppose that $H \subseteq G$ and let N be a normal subgroup of G contained in H. Let $\theta \in \mathrm{Irr}(N)$ and write $T = I_G(\theta)$ for the stabilizer of θ in G. We say that θ is ***H*-good** (with respect to G) if for every $g \in G$ we have that $H^g \cap T$ is contained in some T-conjugate of $H \cap T$.

We recall that if G is a solvable group, then H is a **nilpotent injector** of G whenever $H \cap S$ is a maximal nilpotent subgroup of S for every subnormal

$S \lhd \lhd G$. It turns out that the nilpotent injectors of G are the maximal nilpotent subgroups of G containing $\mathbf{F}(G)$ and that any two of them are G-conjugate ([4]).

(2.1) THEOREM: *Let G be solvable and let N be a nilpotent normal subgroup of G. Let $\theta \in \mathrm{Irr}(N)$ and let $T = I_G(\theta)$ be the stabilizer of θ in G.*

(a) *If X is a nilpotent subgroup of T containing $\mathbf{F}(T)$, then $X\mathbf{F}(G)$ is nilpotent. In particular,*
$$\mathbf{F}(\mathbf{F}(G)T) = \mathbf{F}(G)\mathbf{F}(T).$$

(b) *If J is a nilpotent injector of T, then there exists a nilpotent injector H of G such that $H \cap T = J$. In fact, θ is H-good (with respect to G) for every such H.*

We will prove Theorem (2.1) by applying the main result of [6].

(2.2) THEOREM: *Suppose that $J \subseteq G$ and let $\gamma \in \mathrm{Irr}(J)$ be such that $\gamma^G \in \mathrm{Irr}(G)$. If $|G : J|$ or $|J : \mathbf{F}(J)|$ is odd, then $\mathbf{F}(G)\mathbf{F}(J)$ is nilpotent.*

Proof: This is Theorem A of [6]. ∎

Proof of Theorem (2.1): Suppose that $\mathbf{F}(T) \subseteq X \subseteq T$ is a nilpotent subgroup of T. First we prove that $X\mathbf{F}(G)$ is nilpotent.

Write $F = \mathbf{F}(G)$ and note that
$$N \subseteq F \cap T \subseteq \mathbf{F}(T) \subseteq X.$$

Therefore
$$T \cap FX = X,$$
and X is the stabilizer of θ in FX. Now, if $\gamma \in \mathrm{Irr}(X \mid \theta)$, it follows that γ^{FX} is irreducible by the Clifford correspondence. Since X is nilpotent, we have that $\mathbf{F}(FX)X$ is nilpotent by Theorem (2.2). Hence, FX is nilpotent, as desired. In particular, we deduce that $\mathbf{F}(FT) = F\mathbf{F}(T)$.

Now, let J be a nilpotent injector of T and notice that FJ is a nilpotent injector of FT because it is a maximal nilpotent subgroup containing $\mathbf{F}(FT) = F\mathbf{F}(T)$.

Now, let $FJ \subseteq H \subseteq G$ be a maximal nilpotent subgroup of G and note that H is a nilpotent injector of G. Then $J \subseteq H \cap T$, and by the maximality of J, we conclude that $H \cap T = J$.

Finally, we prove that θ is H-good. It suffices to show that if K is any nilpotent injector of G (that is, if K is any G-conjugate of H), then $K \cap T$ is contained in some T-conjugate of $J = H \cap T$. Now, by Theorem 2.c of [4], we have that $K \cap FT$ is contained in some nilpotent injector of FT. We have already proved that FJ

is a nilpotent injector of FT. Thus there exists $t \in T$ such that $K \cap FT \subseteq FJ^t$. Then

$$K \cap T = K \cap T \cap FT \subseteq FJ^t \cap T = J^t(F \cap T) = J^t,$$

and the proof of the theorem is complete. ∎

The following example due to M. Isaacs shows that the hypothesis of N being nilpotent in Theorem (2.1) is necessary.

(2.3) Example: Let Q be isomorphic to Q_8 and let $U = ES$ be the group of order $2 \cdot 27$ obtained by letting a group S of order 2 act on an extraspecial group E of order 27 and exponent 3 in such a way that S centralizes the center Z of E and inverts all elements of E/Z. Fix a subgroup K of order 9 in U and note that $K \triangleleft U$ and U/K is nonabelian of order 6. Finally, let U act on Q with K in the kernel of the action and U/K acting faithfully. Let $G = QU$ be the semidirect product. Note that $K \triangleleft G$ and G/K can be taken to be isomorphic to $\mathrm{GL}(2,3)$.

Now let L be a subgroup of order 9 in E different from K and write $L = ZY$, where Y has order 3 and is inverted by S. Note that $L \triangleleft U$ so that $QL \triangleleft G$. Write $N = QL$ and $A = QY$ and note that N has index 6 in G. Also, $N = Z \times A$ and A is isomorphic to $\mathrm{SL}(2,3)$ and is normalized by S. Let φ be a faithful irreducible character of A of degree 2 such that φ is stabilized by S. (This is possible because A has exactly three irreducible faithful characters of degree 2.) Define $\theta \in \mathrm{Irr}(N)$ by $\theta = \lambda \times \varphi$, where λ is a nontrivial linear character of Z.

Let T be the stabilizer of θ in G. We claim that $T = NS$. Of course, $N \subseteq T$. To see that S stabilizes θ, it suffices to observe that Z is central in G so that S fixes λ. (We already know that S fixes φ.) We now know that T contains NS and since NS has index 3 in G it suffices to show that $T < G$. In fact, let $x \in K$ with $x \notin Z$ and let y be a generator of Y. Then $y^x = yz$ for some nonidentity element z of Z. Thus $\theta(y^x) = \lambda(z)\theta(y)$ and $\lambda(z) \neq 1$. Also, $\theta(y) = \varphi(y) \neq 0$, and it follows that x does not fix θ.

Now T is the direct product of AS and Z. Thus $\mathbf{F}(T) = QZ$. The group QSZ is thus nilpotent and hence is in a nilpotent injector of T. In particular, S is contained in a nilpotent injector of T. But S cannot be contained in any nilpotent injector of G because otherwise it would centralize the $2'$-part of the Fitting subgroup of G, and yet S does not centralize K.

3. Review of good bases

For the reader's convenience, we review in this section some of the results in [8].

If G is a finite group, we denote by $\mathrm{cf}(G)$ the space of complex class functions defined on G. We fix H a subgroup of G. We let

$$\mathrm{vcf}(G\,|\,H) = \{\alpha \in \mathrm{cf}(G) |\ \alpha(x) = 0 \text{ for } x \in G - \bigcup_{g\in G} H^g\}.$$

It is easy to check (see Lemma (2.1) of [8]) that

$$\mathrm{vcf}(G\,|\,H) = \{\alpha^G |\ \alpha \in \mathrm{cf}(H)\}.$$

Now, let N be a normal subgroup of G contained in H. If $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G\,|\,\theta)$ is the set of irreducible constituents of θ^G. Also, $\mathrm{cf}(G\,|\,\theta)$ is the $\mathbb{C}$-span of the set $\mathrm{Irr}(G\,|\,\theta)$. If Θ is a complete set of representatives of the orbits of the action of G on $\mathrm{Irr}(N)$, then it is clear that

$$\mathrm{cf}(G) = \bigoplus_{\theta\in\Theta} \mathrm{cf}(G\,|\,\theta).$$

We denote

$$\mathrm{vcf}(G\,|\,H,\theta) = \mathrm{vcf}(G\,|\,H) \cap \mathrm{cf}(G\,|\,\theta).$$

(3.1) LEMMA: *Let $N \triangleleft G$ and let $N \subseteq H \subseteq G$. Let Θ be a complete set of representatives of the action of G on $\mathrm{Irr}(N)$. Then*

$$\mathrm{vcf}(G\,|\,H) = \bigoplus_{\theta\in\Theta} \mathrm{vcf}(G\,|\,H,\theta).$$

Proof: This is Lemma (2.2) of [8]. ∎

Next is one of the reasons why H-good characters are important for us.

(3.2) LEMMA: *Suppose that N is a normal subgroup of G contained in $H \subseteq G$, let $\theta \in \mathrm{Irr}(N)$ be H-good and let $T = I_G(\theta)$. Then induction defines an isomorphism $\mathrm{vcf}(T\,|\,T\cap H,\theta) \to \mathrm{vcf}(G\,|\,H,\theta)$.*

Proof: This is Lemma (2.4) of [8]. ∎

A basis $\mathcal{B}$ of $\mathrm{vcf}(G\,|\,H)$ is **good** if it satisfies the following two conditions:

(I) If $\eta \in \mathcal{B}$, then there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = \eta$; and

(D) if $\gamma \in \mathrm{Irr}(H)$, then $\gamma^G = \sum_{\eta\in\mathcal{B}} a_\eta \eta$ for uniquely determined nonnegative integers a_η.

It is easy to show that good bases are necessarily unique (Theorem (2.2) of [5]) and we will denote by $P(G\,|\,H)$ the unique good basis (if it exists) of $\mathrm{vcf}(G\,|\,H)$. Note that $P(G\,|\,H) = P(G\,|\,H^g)$ for every $g \in G$.

It is not in general true that good bases exist for every subgroup H of G. It is straightforward to check that good bases exist whenever $H \triangleleft G$. However, this is already false for $H \triangleleft\triangleleft G$. Perhaps this is a good place to write down an example.

(3.3) Example: Suppose that S is a group of order 2 which interchanges two Klein four groups K. Let $G = (K \times K)S$ be the semidirect product and let $H = \langle x \rangle \times K$ where $1 \neq x \in K$. Then the 8 characters $\{\lambda^G \mid \lambda \in \mathrm{Irr}(H)\}$ are all distinct. By degrees, all of them should be inside $P(G \mid H)$ (if this exists). However, there are only 7 conjugacy classes of G meeting H and it follows that the dimension of the space $\mathrm{vcf}(G \mid H)$ is 7. This is not possible.

Let $N \triangleleft G$ with $N \subseteq H \subseteq G$, and let $\theta \in \mathrm{Irr}(N)$. A basis $\mathcal{B}$ of $\mathrm{vcf}(G \mid H, \theta)$ is **good** if it satisfies the following conditions:

(I) if $\eta \in \mathcal{B}$, then there exists $\alpha \in \mathrm{Irr}(H \mid \theta)$ such that $\alpha^G = \eta$; and

(D) if $\gamma \in \mathrm{Irr}(H \mid \theta)$, then $\gamma^G = \sum_{\eta \in \mathcal{B}} a_\eta \eta$ for uniquely determined nonnegative integers a_η.

The same elementary argument shows that good bases "over" irreducible characters are necessarily unique. We will denote by $P(G \mid H, \theta)$ the unique good basis (if it exists) of $\mathrm{vcf}(G \mid H, \theta)$.

We will need the "Clifford correspondence" for good bases over normal irreducible constituents.

(3.4) LEMMA: *Suppose that $N \triangleleft G$ is contained in $H \subseteq G$. Let $\theta \in \mathrm{Irr}(N)$ be H-good and let $T = I_G(\theta)$. If $P(T \mid T \cap H, \theta)$ is a good basis of $\mathrm{vcf}(T \mid T \cap H, \theta)$, then $\{\eta^G \mid \eta \in P(T \mid T \cap H, \theta)\}$ is a good basis of $\mathrm{vcf}(G \mid H, \theta)$.*

Proof: This is Lemma (2.10) of [8]. ∎

Finally, we will need that good bases exist in the following case. (We refer the reader to [2] for a review of π-theory, the Isaacs set $\mathrm{I}_\pi(G)$ and the definition of Fong characters.)

(3.5) LEMMA: *Let G be a π-separable group and let H be a Hall π-subgroup of G. Let Z be a central π'-subgroup of G and let $\lambda \in \mathrm{Irr}(Z)$. For each $\varphi \in \mathrm{I}_\pi(G)$, let $\alpha_\varphi \in \mathrm{Irr}(H)$ be a Fong character for φ. Then $P(G \mid HZ, \lambda) = \{(\alpha_\varphi \times \lambda)^G \mid \varphi \in \mathrm{I}_\pi(G)\}$. Furthermore,*

$$\bigcup_{\lambda \in \mathrm{Irr}(Z)} P(G \mid HZ, \lambda) = P(G \mid HZ).$$

Proof: The first part easily follows from Theorem (5.3) of [8]. The second part follows from Lemma (2.9) of [8]. ∎

4. Proof of Theorem A

The next result was the main theorem of [5].

(4.1) THEOREM: *Suppose that G is solvable and let H be any nilpotent injector of G. Then there exists the unique basis $P(G\,|\,H)$ of* $\mathrm{vcf}(G\,|\,H)$.

Proof: This is Theorem (3.1) of [5] with the notation of Section 3. ∎

We need to strengthen Theorem (4.1) a bit.

(4.2) THEOREM: *Suppose that G is solvable and let H be any nilpotent injector of G. Let $N \lhd G$ be nilpotent and let $\theta \in \mathrm{Irr}(N)$ be such that $H \cap I_G(\theta)$ is a nilpotent injector of $I_G(\theta)$. Then*

$$P(G\,|\,H) \cap \mathrm{cf}(G\,|\,\theta) = P(G\,|\,H, \theta).$$

Proof: By Theorem (2.1), we may find a complete set of representatives Θ (containing θ) of the action of G on $\mathrm{Irr}(N)$ such that if $\nu \in \Theta$ then $H \cap I_G(\nu)$ is a nilpotent injector of $I_G(\nu)$. If $\eta \in P(G\,|\,H)$, we claim that there exists a unique $\nu \in \Theta$ such that $\eta \in \mathrm{cf}(G\,|\,\nu)$. We know that there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = \eta$. Now, α lies over some ν^g for $\nu \in \Theta$ and $g \in G$. Now, if χ is an irreducible constituent of $\eta = \alpha^G$, then χ lies over α and thus over ν^g. Hence, $\chi \in \mathrm{Irr}(G\,|\,\nu)$ and the claim follows. Now, by Lemma (2.8) of [8], it follows that $\eta = \gamma^G$ for some character γ of H all of whose irreducible constituents lie over ν. Since $\eta \in P(G\,|\,H)$, it follows that $\gamma \in \mathrm{Irr}(H)$ by condition (D) of good bases. We see that there exists a unique $\nu \in \Theta$ such that $\eta = \gamma^G$ for some $\gamma \in \mathrm{Irr}(H\,|\,\nu)$.

Since

$$\mathrm{vcf}(G\,|\,H) = \bigoplus_{\nu \in \Theta} \mathrm{vcf}(G\,|\,H, \nu)$$

by Lemma (3.1), it easily follows that exactly those $\eta \in P(G\,|\,H)$ which lie over ν form the set $P(G\,|\,H, \nu)$. ∎

Before proving Theorem A, recall that $P(G\,|\,H) = P(G\,|\,H^g)$ for every $g \in G$. Hence, the N-blocks of G do not depend on the nilpotent injector that we choose.

This is Theorem A of the introduction.

(4.3) THEOREM: *Let G be a solvable group, let N be a normal nilpotent subgroup of G and let B be an* N*-block of G.*

(i) *There exists $\theta \in \mathrm{Irr}(N)$ such that $B \subseteq \mathrm{Irr}(G\,|\,\theta)$.*

(ii) *If T is the stabilizer of θ in G, then there exists an* N*-block b of T such that $b \subseteq \mathrm{Irr}(T\,|\,\theta)$ and*

$$B = \{\psi^G \,|\, \psi \in b\}.$$

Proof: Let B be an N-block of G and fix H a nilpotent injector of G. To show that all characters of B lie over some $\theta \in \mathrm{Irr}(N)$, it suffices to prove this fact for any two characters in B which are N-linked. Hence, assume that $\chi, \psi \in \mathrm{Irr}(G)$ are N-linked. Then there exists $\eta \in P(G \mid H)$ such that

$$[\chi, \eta] \neq 0 \neq [\psi, \eta].$$

Now, by definition of the good basis $P(G \mid H)$, there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = \eta$. Since N is normal and nilpotent, we have that $N \subseteq H$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of α_N. Now, since

$$0 \neq [\chi, \eta] = [\chi, \alpha^G] = [\chi_H, \alpha],$$

we have that χ lies over θ. For the same reason, ψ lies over θ. This proves part (i).

Let $T = I_G(\theta)$. By Theorem (2.1) (and replacing θ by some G-conjugate if necessary), we may assume that $T \cap H$ is an injector of T and that θ is H-good.

Now, let $\chi \in B \subseteq \mathrm{Irr}(G \mid \theta)$ and, by the Clifford correspondence, let $\psi \in \mathrm{Irr}(T \mid \theta)$ be such that $\psi^G = \chi$. Let b be the N-block of ψ. By the first part, notice that $b \subseteq \mathrm{Irr}(T \mid \theta)$. We prove that $B = \{\psi^G \mid \psi \in b\}$.

First, note that the elements in $P(G \mid H)$ which are used to define N-linking between the elements of B necessarily lie in $\mathrm{cf}(G \mid \theta)$. By Theorem (4.2), the N-linking between the elements of B are made by the elements in $P(G \mid H, \theta)$. By the same reason, the N-linking between the elements of b are made by the elements in $P(T \mid T \cap H, \theta)$. By Lemma (3.4), we have that

$$\{\eta^G \mid \eta \in P(T \mid T \cap H, \theta)\} = P(G \mid H, \theta).$$

Since $[\mu^G, \tau^G] = [\tau, \mu]$ for $\tau, \mu \in \mathrm{cf}(T \mid \theta)$ by the Clifford correspondence, the proof of Theorem A easily follows. ∎

5. Characters of central products

We shall need the following elementary result.

(5.1) Lemma: *Suppose that $H_1, \ldots, H_n$ are subgroups of G and let $Z = H_1 \cap \cdots \cap H_n$. Assume that $[H_i, H_j] = 1$ for $i \neq j$ and that*

$$G/Z = H_1/Z \times \cdots \times H_n/Z$$

is a direct product. Let $\lambda \in \mathrm{Irr}(Z)$ and let $\theta_i \in \mathrm{Irr}(H_i \mid \lambda)$ for $1 \leq i \leq n$. Then there exists a unique $\chi \in \bigcap_{i=1}^n \mathrm{Irr}(G \mid \theta_i)$. In fact,

$$\chi(h_1 \cdots h_n) = \theta_1(h_1) \cdots \theta_n(h_n)$$

for $h_i \in H_i$.

If $\mu\colon X \to Y$ is a group isomorphism, recall that $\mathrm{Irr}(Y) = \{\chi^\mu \mid \chi \in \mathrm{Irr}(X)\}$, where $\chi^\mu(x^\mu) = \chi(x)$ for $x \in X$.

Proof of Lemma (5.1): Of course, we may assume that $n \geq 2$. Consider the map $\tau\colon H_1 \times \cdots \times H_n \to G$ given by $(h_1, \ldots, h_n)^\tau = h_1 \cdots h_n$. Note that τ is a surjective group homomorphism. Let $N = \ker(\tau)$. Call μ the group isomorphism $H_1 \times \cdots \times H_n/N \to G$ induced by τ. Since $Z \subseteq \mathbf{Z}(G)$ because $n \geq 2$, note that $(\theta_i)_Z = \theta_i(1)\lambda$.

Suppose that $\psi \in \mathrm{Irr}(G)$ lies over θ_i for every i. Let $\xi \in \mathrm{Irr}(H_1 \times \cdots \times H_n/N)$ be such that $\xi^\mu = \psi$. Then $\xi = \gamma_1 \times \cdots \times \gamma_n$ for some $\gamma_i \in \mathrm{Irr}(H_i)$, and we deduce that $\psi(h_1 \cdots h_n) = \gamma_1(h_1) \cdots \gamma_n(h_n)$. Since ψ_{H_i} is a multiple of θ_i, we easily deduce that $\gamma_i = \theta_i$ for each i. Therefore, $\psi(h_1 \cdots h_n) = \theta_1(h_1) \cdots \theta_n(h_n)$ and we see that if such a ψ exists, then it is unique.

Now, let $\alpha = \theta_1 \times \cdots \times \theta_n$. It suffices to show that $N \subseteq \ker(\alpha)$. Suppose that $(h_1, \ldots, h_n) \in N$ which happens if and only if $h_1 \cdots h_n = 1$. Then $(h_1 Z) \cdots (h_n Z) = Z$, and we deduce that $h_i \in Z$ for all i. Now,

$$\begin{aligned}\alpha(h_1, \ldots, h_n) &= \theta_1(h_1) \cdots \theta_n(h_n) = \theta_1(1) \cdots \theta_n(1)\lambda(h_1) \cdots \lambda(h_n) \\ &= \alpha(1)\lambda(h_1 \cdots h_n) = \alpha(1).\end{aligned}$$

Now, $\chi = \alpha^\mu$ satisfies the conclusions of the lemma. ∎

Under the hypothesis and notation of Lemma (5.1), we will write

$$\chi = \theta_1 \wedge \cdots \wedge \theta_n = \bigwedge_{i=1}^n \theta_i.$$

6. Proof of Theorems B and C

In several proofs throughout this paper, we will distinguish two cases, according to whether or not the nilpotent injector is contained in some proper normal subgroup of G.

(6.1) LEMMA: *Suppose that H is a nilpotent injector of a solvable group G and assume that $H \subseteq M \lhd G$. Let $P(M \mid H)$ be the good basis of* $\mathrm{vcf}(M \mid H)$ *and let $\mathcal{C}$ be a complete set of representatives of the orbits of $\mathbf{N}_G(H)$ on its action on $P(M \mid H)$. Then $\{\eta^G \mid \eta \in \mathcal{C}\} = P(G \mid H)$.*

Proof: The proof of this lemma follows by Lemma (3.2) of [5] and the uniqueness of good bases. (The fact that $G = M\mathbf{N}_G(H)$ in this case, follows from the Frattini argument.) ∎

We are going to use the next lemma several times.

(6.2) LEMMA: *Let H be a π-subgroup of G and let Z be a central π'-subgroup of G. If $\alpha \in \mathrm{Irr}(H)$ and $\lambda \in \mathrm{Irr}(Z)$, then*

$$((\alpha \times \lambda)^G)_{HZ} = \frac{1}{|Z|}((\alpha^G)_H \times \lambda).$$

Proof: Let $hz \in HZ = H \times Z$, where $h \in H$ and $z \in Z$. Then

$$(\alpha \times \lambda)^G(hz) = \frac{1}{|HZ|} \sum_{\substack{g \in G \\ ghzg^{-1} \in HZ}} (\alpha \times \lambda)(ghzg^{-1}).$$

Now, by using that Z is central and the uniqueness of the π-π' decomposition of elements, notice that $g(hz)g^{-1} \in HZ$ if and only if $ghg^{-1} \in H$. Hence,

$$\begin{aligned}(\alpha \times \lambda)^G(hz) &= \frac{1}{|HZ|} \sum_{\substack{g \in G \\ ghg^{-1} \in H}} \alpha(ghg^{-1})\lambda(z) \\ &= \frac{1}{|Z|}(\alpha^G(h)\lambda(z)) = \frac{1}{|Z|}((\alpha^G)_H \times \lambda)(hz),\end{aligned}$$

as required. ∎

We are going to use the construction of the nilpotent injectors due to A. Mann ([4]). Suppose that G is solvable and let F be the Fitting subgroup of G. For every prime p dividing $|F|$, let $F_{p'}$ be the p-complement of F and let S_p be any Sylow p-subgroup of $\mathbf{C}_G(F_{p'})$. Then $[S_p, S_q] = 1$ for $p \neq q$ and

$$\prod_{p \,|\, |F|} S_p$$

is a nilpotent injector of G.

If H is a subgroup of G, we denote by H^G the normal closure of H in G.

(6.3) LEMMA: *Suppose that G is a solvable group and assume that H is a nilpotent injector of G with $H^G = G$. For every prime p dividing $|H|$, let $N_p = (H_p)^G\mathbf{Z}(G)$, where H_p is the Sylow p-subgroup of H. Then H_p is a Sylow p-subgroup of N_p, $[N_p, N_q] = 1$ for $p \neq q$ and*

$$G/\mathbf{Z}(G) = \prod_p N_p/\mathbf{Z}(G)$$

is a direct product.

Proof: This is Lemma (3.3) of [5]. ∎

(6.4) LEMMA: *Let G be a solvable group and assume that H is a nilpotent injector of G with $H^G = G$. Let $\lambda \in \mathrm{Irr}(Z)$, where $Z = \mathbf{Z}(G)$. Let $\alpha_p, \beta_p \in \mathrm{Irr}(N_p \,|\, \lambda)$. Using the notation of Lemma (6.3), we have that $\bigwedge_p \alpha_p$ and $\bigwedge_p \beta_p$ lie in the same* N*-block of G if α_p and β_p lie in the same* N*-block of N_p for every p.*

Proof: First, note that every irreducible character which is N-linked to α_p necessarily lies over λ (by Theorem A, for instance). By repeating terms, we may assume that α_p and β_p are N-linked by a chain of characters of length not depending on p. By using this fact, note that it suffices to show that if α_p and β_p are N-linked for each prime p, then $\bigwedge_p \alpha_p$ and $\bigwedge_p \beta_p$ are N-linked.

We claim that $H_p \times Z_{p'}$ is a nilpotent injector of N_p, where $Z_{p'}$ is the p-complement of Z. We know that $H \cap N_p$ is a nilpotent injector of $N_p \lhd G$. Also, $H \cap N_p = H_p(H_{p'} \cap N_p)$. Now, since $\prod_p N_p/Z$ is a direct product, we have that $H_{p'} \cap N_p \subseteq Z_{p'}$. Since it is clear that $Z_{p'} \subseteq H_{p'} \cap N_p$, the claim follows.

Note that by Lemma (5.1) we have that

$$(\bigwedge_p \alpha_p)_H = \prod_p (\alpha_p)_{H_p}.$$

Suppose now that $\gamma_p \in \mathrm{Irr}(H_p)$. Now,

$$(1)\quad [\bigwedge_p \alpha_p, (\prod_p \gamma_p)^G] = [(\bigwedge_p \alpha_p)_H, \prod_p \gamma_p] = [\prod_p (\alpha_p)_{H_p}, \prod_p \gamma_p] = \prod_p [(\alpha_p)_{H_p}, \gamma_p].$$

By Lemma (3.5) (using that H_p is a Sylow p-subgroup of N_p and the uniqueness of good bases), we have that there exists $\gamma_p \in \mathrm{Irr}(H_p)$ which is p-Fong in N_p, such that α_p and β_p are irreducible constituents of $(\gamma_p \times \lambda_{p'})^{N_p}$, where $\lambda_{p'}$ is the p'-part of λ. Now, by Theorem (3.6) of [5], it suffices to show that

$$[\bigwedge_p \alpha_p, (\prod_p \gamma_p)^G] \neq 0 \neq [\bigwedge_p \beta_p, (\prod_p \gamma_p)^G].$$

Since α_p lies over $\gamma_p \times \lambda_{p'}$, in particular, we have that α_p lies over γ_p. Hence $[(\alpha_p)_{H_p}, \gamma_p] \neq 0$, and by (1) we deduce that $\bigwedge_p \alpha_p$ and $\bigwedge_p \beta_p$ are N-linked. ∎

We are ready to prove Theorem B.

(6.5) THEOREM: *Let G be a solvable group, let $F = \mathbf{F}(G)$ and let $\theta \in \mathrm{Irr}(F)$ be G-invariant. Then $\mathrm{Irr}(G \mid \theta)$ is an* N*-block of G.*

Proof: We argue by induction on $|G|$. Let $\chi, \psi \in \mathrm{Irr}(G \mid \theta)$. We wish to prove that χ and ψ are connected by a chain of N-linked characters.

Let H be a nilpotent injector of G and assume that $H \subseteq M \lhd G$, where $|M| < |G|$. By Lemma (6.1), note that if $\tau, \mu \in \mathrm{Irr}(M)$ are N-linked, then every irreducible constituent of τ^G is N-linked with every irreducible constituent of μ^G. Since $F = \mathbf{F}(M)$ and θ is M-invariant, the theorem easily follows by induction in this case.

From now on, we assume that $H^G = G$ and we use the notation of Lemma (6.3) and Lemma (6.4). Write $Z = \mathbf{Z}(G)$ and let $\lambda \in \mathrm{Irr}(Z)$ be the unique irreducible constituent of θ_Z. Assume that $N_p < G$ for every prime p dividing $|F|$. Let χ_p be the unique irreducible constituent of χ_{N_p} and let ψ_p be the unique irreducible constituent of ψ_{N_p}. Using the notation of Section 5, note that $\chi = \bigwedge_p \chi_p$ and $\psi = \bigwedge_p \psi_p$.

By using Lemma (6.3), it is clear that $\mathbf{F}(N_p) = F_p \times Z_{p'}$, where $Z_{p'}$ is the p-complement of Z and F_p is the Sylow p-subgroup of F. Also, $H_p \times Z_{p'}$ is a nilpotent injector of N_p (see the second paragraph of the proof of Lemma (6.4)).

Write $\theta = \prod \theta_p$, where $\theta_p \in \mathrm{Irr}(F_p)$. Now, notice that each χ_p and ψ_p lie over θ_p because χ and ψ lie over θ, and thus, over θ_p. Also, if $\lambda_{p'}$ is the p'-part of λ, note that χ_p and ψ_p lie over $\lambda_{p'}$ because χ and ψ lie over λ and thus over $\lambda_{p'}$. Now, $\chi_p(hz) = \lambda_{p'}(z)\chi_p(h)$ for $h \in H_p$ and $z \in Z_{p'}$ and it easily follows that χ_p (and ψ_p) lie over $\theta_p \times \lambda_{p'}$. Since $\theta_p \times \lambda_{p'}$ is G-invariant, by induction we deduce that χ_p and ψ_p lie in the same N-block for every prime p. In this case, by applying Lemma (6.4), we are done.

We may assume, therefore, that $N_p = G$ for some prime p dividing $|F|$. In this case, the nilpotent injectors of G are $P \times Z_{p'}$, where P is a Sylow p-subgroup of G. Also, $F = F_p \times Z_{p'}$.

Now, we apply the main results of [9]. If $\delta, \xi \in \mathrm{Irr}(G)$, we write $\delta \leftrightarrow_p \xi$ if there exists a Fong character $\alpha \in \mathrm{Irr}(P)$ such that δ and ξ are irreducible constituents of α^G. (By Corollary (2.5) of [2], this provides the same p-linking as defined by M. Slattery in Section 2 of [9].) Since χ and ψ lie over θ_p, $F_p = \mathbf{O}_p(G)$ and θ_p is G-invariant, by Theorem (2.8) of [9], it follows that there exists a chain of irreducible characters $\tau_i \in \mathrm{Irr}(G)$

$$\chi = \tau_0 \leftrightarrow_p \tau_1 \leftrightarrow_p \cdots \leftrightarrow_p \tau_s \leftrightarrow_p \tau_{s+1} = \psi$$

and Fong characters $\alpha_i \in \mathrm{Irr}(P)$ such that τ_i and τ_{i+1} are irreducible constituents of α_i^G for $0 \leq i \leq s$. Thus

$$[\alpha_i^G, \alpha_{i+1}^G] \neq 0 \quad \text{for } 0 \leq i \leq s-1.$$

Now, by applying Lemma (6.2), we have that

$$[(\alpha_i \times \lambda_{p'})^G, (\alpha_{i+1} \times \lambda_{p'})^G] = [((\alpha_i \times \lambda_{p'})^G)_{P \times Z_{p'}}, \alpha_{i+1} \times \lambda_{p'}]$$
$$= \frac{1}{|Z_{p'}|}[(\alpha_i^G)_P \times \lambda_{p'}, \alpha_{i+1} \times \lambda_{p'}] = \frac{1}{|Z_{p'}|}[(\alpha_i^G)_P, \alpha_{i+1}] = \frac{1}{|Z_{p'}|}[\alpha_i^G, \alpha_{i+1}^G] \neq 0$$

for $0 \leq i \leq s-1$. Now, we have that there exist common irreducible constituents $\xi_i \in \mathrm{Irr}(G)$ of $(\alpha_i \times \lambda_{p'})^G$ and $(\alpha_{i+1} \times \lambda_{p'})^G$ for $0 \leq i \leq s-1$. Also, recall that $\chi = \tau_0$ lies over α_0 and $\lambda_{p'}$, and therefore, χ lies over $\alpha_0 \times \lambda_{p'}$. By the same argument, ψ is an irreducible constituent of $(\alpha_s \times \lambda_{p'})^G$. Now (using Lemma (3.5)), we deduce that $\chi, \xi_0, \xi_1, \ldots, \xi_{s-1}, \psi$ is a chain of N-linked characters. This proves the theorem. ∎

Next is Theorem C of the introduction. Of course, it is the analog of the k(B)-conjecture for N-blocks.

(6.6) THEOREM: *Suppose that G is a solvable group, let B be an* N-*block of G and let H be a nilpotent injector of G. Then*

$$|B| \leq |G : H|.$$

Proof: We argue by induction on $|G|$.

Let $F = \mathbf{F}(G)$. By Theorem A, there exists $\theta \in \mathrm{Irr}(F)$ and an N-block b of $T = I_G(\theta)$ such that $B \subseteq \mathrm{Irr}(G \,|\, \theta)$, $b \subseteq \mathrm{Irr}(T \,|\, \theta)$ and $B = \{\psi^G \,|\, \psi \in b\}$. By the uniqueness in the Clifford correspondence, note that $|B| = |b|$.

Let J be a nilpotent injector of T. By Theorem (2.1), there exists a G-conjugate K of H such that $K \cap T = J$.

Assume first that $T < G$. Then, by the inductive hypothesis, we will have that

$$|B| = |b| \leq |T : J| = |T : K \cap T| \leq |G : K| = |G : H|,$$

as desired.

Hence, we may assume that $T = G$. In this case, by Theorem B we have that $B = \mathrm{Irr}(G \,|\, \theta)$. Then $|B| \leq |G : H|$ by Corollary B of [1]. ∎

7. Nilpotent injectors in the character table

If K is a conjugacy class of G and π is a set of primes, then K_π denotes the conjugacy class of x_π, where x is any element of K.

(7.1) LEMMA: *If K is a conjugacy class of G, then the character table of G uniquely determines K_π.*

Proof: Let ρ be the set of prime divisors of the order of $x \in K$. By Higman's theorem (8.21) of [3], we know that the character table of G determines ρ. Certainly, we may assume that $\pi \subseteq \rho$.

We argue by induction of $|\rho|$. If $|\rho| = 0$, then $x = 1$ and the lemma is trivially true. If $\pi = \rho$, then $K_\pi = K_\rho = K$ and the lemma is also true. So, we may find a prime $p \in \rho - \pi$. Let $\sigma = \rho - \{p\}$. If $L = K_\sigma$, then $L_\pi = K_\pi$. By induction, therefore, it suffices to show that the character table of G determines K_σ.

Now, let $p\mathbf{R} \subseteq M \subseteq \mathbf{R}$ be a maximal ideal of the ring of algebraic integers $\mathbf{R}$. Let T be any conjugacy class of p'-elements of G (all of them are determined by the character table of G) and let $y \in T$. By Theorem (8.20) of [3], we know that $T = K_\sigma$ if and only if

$$\chi(x) \equiv \chi(y) \bmod M$$

for every $\chi \in \mathrm{Irr}(G)$. Since the latter equation can be determined from the character table, the proof of the lemma is complete. ∎

(7.2) LEMMA: *Let K be the conjugacy class of $x \in G$, where x is a π'-element. Let N be a normal π-subgroup of G. Then*

$$K \subseteq \mathbf{C}_G(N) \quad \text{iff} \quad \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} |\chi(x)|^2 = \frac{\sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2}{|N|}.$$

Proof: By using the orthogonality relations, we have to prove that

$$K \subseteq \mathbf{C}_G(N) \quad \text{iff } |\mathbf{C}_{G/N}(xN)| = \frac{|\mathbf{C}_G(x)|}{|N|}.$$

Assume first that

$$|\mathbf{C}_{G/N}(xN)| = \frac{|\mathbf{C}_G(x)|}{|N|}.$$

Then

$$\frac{|\mathbf{C}_G(x)|}{|\mathbf{C}_N(x)|} = |\mathbf{C}_G(x)/\mathbf{C}_N(x)| = |\mathbf{C}_G(x)N/N| \leq |\mathbf{C}_{G/N}(xN)| = \frac{|\mathbf{C}_G(x)|}{|N|},$$

and we deduce that $\mathbf{C}_N(x) = N$.

Conversely, assume that $N \subseteq \mathbf{C}_G(x)$. Then $yN \in \mathbf{C}_{G/N}(xN)$ if and only if $x^yN = xN$ if and only if $x^y = xn$ for some $n \in N$. Since $[x, N] = 1$, x is a π'-element and N is a π-group, we deduce that $yN \in \mathbf{C}_{G/N}(xN)$ if and only if $x^y = x$. Hence

$$\mathbf{C}_{G/N}(xN) = \mathbf{C}_G(x)/N,$$

and the proof of the lemma is complete. ∎

(7.3) COROLLARY: *Let N be a normal π-subgroup of G. Then the character table of G determines $\mathbf{O}^{\pi}(\mathbf{C}_G(N))$.*

Proof: Let $K_1, \ldots, K_t$ be the conjugacy classes of G consisting of π'-elements centralizing N. (These are determined by the character table of G by applying Higman's theorem (8.21) and Lemma (7.2).) Let

$$M = \bigcap_{\substack{L \triangleleft G, \\ K_i \subseteq L}} L = \langle K_1, \ldots, K_t \rangle.$$

We claim that M (which is determined by the character table of G) is exactly $\mathbf{O}^{\pi}(\mathbf{C}_G(N))$. Since $K_i \subseteq \mathbf{O}^{\pi}(\mathbf{C}_G(N))$, it is clear that $M \subseteq \mathbf{O}^{\pi}(\mathbf{C}_G(N))$. On the other hand, if L is a conjugacy class of $\mathbf{C}_G(N)$ consisting of π'-elements, then L is contained in some K_i. Therefore, M contains all conjugacy classes of $\mathbf{C}_G(N)$ consisting of π'-elements. Hence $\mathbf{C}_G(N)/M$ is a π-group and therefore

$$\mathbf{O}^{\pi}(\mathbf{C}_G(N)) \subseteq M,$$

as desired. ∎

Next is Theorem D of the introduction.

(7.4) COROLLARY: *Let G be a solvable group and let H be a nilpotent injector of G. Then the character table of G determines the set*

$$\bigcup_{g \in G} H^g$$

and $|H|$.

Proof: We know that the character table of G determines $\mathbf{O}_p(G)$ for every prime p. Hence, it also determines $F_{p'}$, the p-complement of $F = \mathbf{F}(G)$. By Corollary (7.3), it also determines $\mathbf{O}^{p'}(\mathbf{C}_G(F_{p'}))$. Since

$$|H| = \prod_{p \mid |F|} |\mathbf{O}^{p'}(\mathbf{C}_G(F_{p'}))|_p,$$

the second part easily follows. Also, since every conjugacy class of G meets H if and only if $K_p \subseteq \mathbf{O}^{p'}(\mathbf{C}_G(F_{p'}))$, the proof of the corollary is complete by applying Lemma (7.1) and (7.3). ∎

8. Proof of Theorem E

This is Theorem E of the introduction.

(8.1) THEOREM: *Let H be a nilpotent injector of a solvable group G. Suppose that λ and μ are linear characters of H. Then $\lambda^G = \mu^G$ if and only if $\lambda = \mu^x$ for some $x \in \mathbf{N}_G(H)$.*

Proof: We argue by induction on $|G|$. Write $F = \mathbf{F}(G)$.

By degrees and condition (D) of good bases, notice that $\lambda^G \in P(G \mid H)$.

Suppose that $H \subseteq M \triangleleft G$. Then $G = M\mathbf{N}_G(H)$ by the Frattini argument. Also, $(\lambda^G)_M = ((\lambda^M)^G)_M$ is a sum of $\mathbf{N}_G(H)$-conjugates of λ^M by Mackey's theorem. By the same argument, $(\mu^G)_M$ is a sum of $\mathbf{N}_G(H)$-conjugates of μ^M. By the second paragraph of this proof, note that λ^M and μ^M (and therefore every $\mathbf{N}_G(H)$-conjugate) lie in $P(M \mid H)$. Since $(\lambda^G)_M = (\mu^G)_M$, by the linear independence of $P(M \mid H)$ it follows that $\lambda^M = (\mu^M)^x$ for some $x \in \mathbf{N}_G(H)$. Now,

$$\lambda^M = (\mu^x)^M$$

and by induction we deduce that λ and μ^x are $\mathbf{N}_M(H)$-conjugate. This proves the theorem in this case.

So we may assume that $H^G = G$. We use the notation of Lemma (6.3). Now, write

$$\lambda = \prod_p \lambda_p,$$

where $\lambda_p \in \mathrm{Irr}(H_p)$ is the p-part of λ for the primes p dividing $|F|$. Also, write

$$\mu = \prod_p \mu_p.$$

By Lemma (3.4) of [5], we have that

$$((\prod_p \lambda_p)^G)_H = \frac{1}{|Z|^{n-1}} \prod_p (\lambda_p^{N_p})_{H_p},$$

where n is the number of different primes dividing $|F|$. Since $\lambda^G = \mu^G$, we deduce that

$$(\lambda_p^{N_p})_{H_p} = (\mu_p^{N_p})_{H_p}.$$

Then

$$\lambda_p^{N_p} = \mu_p^{N_p}$$

for every p dividing $|F|$. Since H_p is a Sylow p-subgroup of N_p, by Isaacs' Corollary (6.1) of [2], we deduce that there exists $x_p \in \mathbf{N}_{N_p}(H_p)$ such that

$$(\lambda_p)^{x_p} = \mu_p.$$

Since $[N_p, N_q] = 1$ for $q \neq p$, notice that

$$\prod_p x_p \in \mathbf{N}_G(H)$$

is such that

$$\lambda^{\prod_p x_p} = \mu,$$

as desired. ∎

9. Final remarks

If H is a nilpotent injector of a solvable group G, put

$$G^0 = \bigcup_{g \in G} H^g$$

and let $\mathrm{cf}(G^0)$ be the space of complex class functions $G^0 \to \mathbb{C}$.

Is there some canonical basis of $\mathrm{cf}(G^0)$? In this section we discuss this and some other natural questions.

First of all, it is clear that

$$\dim(\mathrm{cf}(G^0)) = \dim(\mathrm{vcf}(G \mid H)).$$

If $\varphi, \theta \in \mathrm{cf}(G^0) \cup \mathrm{cf}(G)$, we write

$$[\varphi, \theta]^0 = \frac{1}{|G|} \sum_{x \in G^0} \varphi(x)\overline{\theta(x)}.$$

Note that

$$[\cdot, \cdot]^0 \colon \mathrm{cf}(G^0) \times \mathrm{vcf}(G \mid H) \to \mathbb{C}$$

is a complex hermitian bilinear form. Furthermore, this form is non-degenerate. (See Section 3 of [8].) It follows that given the basis $P(G \mid H) = \{\eta_1, \dots, \eta_k\}$ there exists a unique basis $I(G \mid H) = \{\varphi_1, \dots, \varphi_k\}$ of $\mathrm{cf}(G^0)$ satisfying

$$[\varphi_i, \eta_j]^0 = \delta_{i,j}.$$

If $\varphi \in I(G \mid H)$, then we denote by Φ_φ the unique element in $P(G \mid H)$ such that

$$[\Phi_\varphi, \mu]^0 = \delta_{\varphi,\mu}$$

for $\mu \in I(G \mid H)$.

If $\chi \in \mathrm{cf}(G)$, let us denote by χ^0 the restriction of χ to G^0.

(9.1) THEOREM: *Suppose that G is solvable and let H be a nilpotent injector of G. If χ is a character of G, then*

$$\chi^0 = \sum_{\varphi \in I(G \mid H)} d_{\chi\varphi}\varphi$$

for uniquely determined nonnegative integers $d_{\chi\varphi}$. Furthermore,

$$d_{\chi\varphi} = [\Phi_\varphi, \chi].$$

Proof: This is Theorem (3.1) and Lemma (3.2) of [8]. ∎

In some sense, it is reasonable to view the basis $I(G \mid H)$ as the set of "irreducible Brauer characters" of G with respect to H, being the integers $d_{\chi\varphi}$, the "decomposition numbers," and the elements of $P(G \mid H)$ as the "projective indecomposable characters."

First, we see that there is no possible "Fong–Swan" theorem for $I(G \mid H)$.

(9.2) Example: Suppose that V is the direct product of two groups of order 3. Let D be the dihedral group of order 8 and center Z. Then D/Z acts faithfully on V. Let $G = VD$ be the semidirect product. In this case, $F = \mathbf{F}(G) = V \times Z$ is also the nilpotent injector of G. Now, let $\theta = 1_V \times \lambda$, where $1 \neq \lambda \in \mathrm{Irr}(Z)$. It is easy to check that $\mathrm{Irr}(G \mid \theta) = \{\chi\}$, where χ is the unique extension of the unique nonlinear character of D containing V in its kernel. It is easy to check that $\theta \in I(G \mid H)$ is not liftable.

For $\varphi, \mu \in I(G \mid H)$, we define the Cartan invariants as

$$c_{\varphi\mu} = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi} d_{\chi\mu} = [\Phi_\varphi, \Phi_\mu].$$

(9.3) THEOREM: *Suppose that G is solvable and let H be a nilpotent injector of G.*

(a) *The matrix*

$$D = (d_{\chi\varphi})_{\chi \in \mathrm{Irr}(G), \varphi \in I(G \mid H)}$$

has rank $|I(G \mid H)|$. Hence, $C = D^t D$ is invertible.

(b) *If $\Gamma = (\gamma_{\varphi\theta})_{\varphi,\theta \in I(G \mid H)} = C^{-1}$ and $\chi, \psi \in \mathrm{Irr}(G)$, then*

$$[\chi, \psi]^0 = \sum_{\varphi,\theta \in I(G \mid H)} [\Phi_\varphi, \chi][\Phi_\theta, \psi]\gamma_{\varphi\theta}.$$

Proof: The matrix

$$D = (d_{\chi\varphi})_{\chi \in \mathrm{Irr}(G), \varphi \in I(G \mid H)}$$

is the matrix of the linear surjective restriction map ^{0}: $\mathrm{cf}(G) \to \mathrm{cf}(G^0)$ with respect to the bases $\mathrm{Irr}(G)$ and $I(G \mid H)$. Then part (a) follows by elementary linear algebra.

Now, write $C = (c_{\varphi\theta})_{\varphi,\theta \in I(G \mid H)}$. Then

$$\begin{aligned}\delta_{\varphi\theta} = [\varphi, \Phi_\theta]^0 &= \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\theta}[\varphi, \chi^0]^0 \\ &= \sum_{\chi \in \mathrm{Irr}(G)} \sum_{\mu \in I(G \mid H)} d_{\chi\theta} d_{\chi\mu}[\varphi, \mu]^0 = \sum_{\mu \in I(G \mid H)} [\varphi, \mu]^0 c_{\mu\theta}.\end{aligned}$$

This proves that

$$\Gamma = ([\varphi, \theta]^0)_{\varphi,\theta \in I(G \mid H)}.$$

Part (b) easily follows from this. ∎

If $\chi, \psi \in \mathrm{Irr}(G)$, note that

$$[\chi, \psi]^0 = \frac{1}{|G|} \sum_{x \in G^0} \chi(x)\overline{\psi(x)}$$

can be read off from the character table of G by Corollary (7.4). It is natural to study the graph associated to the linking

$$\chi \leftrightarrow \psi \quad \text{iff } [\chi, \psi]^0 \neq 0.$$

If G^0 is the set of p-regular elements of any finite group G (where p is a prime), then the connected components of this graph are the Brauer p-blocks of G. (See Theorem (3.19) of [7].) Also, if G^0 is the set of π-elements of a π-separable group G, then the connected components are the Isaacs–Slattery π-blocks. (See Theorem (2.2) of [9].)

In our case, it is not true that the connected components of the graph are the N-blocks, and we will provide an example below.

Suppose that B is an N-block of G and let H be a nilpotent injector of G. We know that χ and ψ are N-linked if and only if there exists $\varphi \in I(G \mid H)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$. We may partition $I(G \mid H)$ into blocks. If B is an N-block, then $I(B \mid H) = \{\varphi \in I(G \mid H) | \ d_{\chi\varphi} \neq 0 \text{ for some } \chi \in B\}$. Notice that the decomposition matrix has a diagonal block form if we arrange the ordinary characters and the elements of $I(G \mid H)$ in N-blocks. Furthermore, C (and Γ) have also diagonal block form.

(9.4) LEMMA: *Let $\chi, \psi \in \mathrm{Irr}(G)$. If $[\chi, \psi]^0 \neq 0$, then χ and ψ lie in the same N-block of G.*

Proof: By Theorem (9.3.b), we have that there exist $\varphi, \theta \in I(G \mid H)$ such that

$$d_{\chi\varphi} d_{\psi\theta} \gamma_{\varphi\theta} \neq 0.$$

Hence, φ and θ lie in the same N-block B of G by the comments preceding the statement of this lemma. Hence, $\chi, \psi \in B$ and the proof of the lemma is complete. ∎

(9.5) Example: Let $S = \mathrm{SL}(2,3)$ and $Z = Z(\mathrm{SL}(2,3))$. Now, suppose that S/Z acts faithfully on some GF(3)-module V. Let $G = VS$ be the semidirect product. Note that $\mathbf{F}(G) = F = V \times Z$. Also, the nilpotent injectors of G are of the form $P \times Z$, where $P \in \mathrm{Syl}_3(G)$. Hence $G^0 = \bigcup_{g \in G} P^g Z$ are the elements of G whose 2-part is in Z.

Now, let $1 \neq \lambda \in \mathrm{Irr}(Z)$ and let $\theta = 1_V \times \lambda$. Note that θ is G-invariant, and therefore, that $\mathrm{Irr}(G \mid \theta)$ is an N-block by Theorem B.

It is clear that restriction of characters defines a bijection

$$\mathrm{Irr}(G \mid \theta) \to \mathrm{Irr}(S \mid \lambda).$$

Recall that $\mathrm{Irr}(S \mid \lambda) = \{\chi_1, \chi_2, \chi_3\}$. These are faithful characters of S of degree 2 vanishing on the elements of S of order 4. Now, let $\mathrm{Irr}(G \mid \theta) = \{\widehat{\chi_1}, \widehat{\chi_2}, \widehat{\chi_3}\}$ be their respective extensions.

We claim that each $\widehat{\chi_i}$ vanishes off G^0. To see this, let $g \in G - G^0$. Consider the isomorphism $\varphi : S \to G/V$ given by $s \mapsto sV$. Suppose that $o(g_2 V) = 2$. Then $g_2 V = \varphi(z) = zV$, where z is the unique element of order 2 of S. Then $g_2 = zv$ for some $v \in V$. Hence $g = g_2 g_3 = zvg_3$. Since $vg_3 \in V\langle g_3 \rangle$ is contained in some Sylow 3-subgroup of G, we will conclude that $g \in G^0$, a contradiction. Therefore, we have that $o(g_2 V) = 4$. Hence, $o(gV) = 4$ since there are not any other type of elements in SL(2,3) whose 2-part is 4. Hence, we may write $gV = sV$, where $s \in S$ has order 4. Now, $\widehat{\chi_i}(g) = \widehat{\chi_i}(gV) = \widehat{\chi_i}(sV) = \chi_i(s) = 0$, as claimed.

Finally, suppose that $\widehat{\chi_i}$ is $\leftrightarrow$-linked to some $\chi \in \mathrm{Irr}(G)$. Assume that $\widehat{\chi_i} \neq \chi$. By Lemma (9.4) and Theorem B, we have that $\chi \in \mathrm{Irr}(G \mid \theta)$. Hence, $\chi = \widehat{\chi_j}$, for some $j \neq i$. Now, by the last paragraph, we have that

$$0 \neq [\widehat{\chi_i}, \widehat{\chi_j}]^0 = [\widehat{\chi_i}, \widehat{\chi_j}] = 0.$$

This is a contradiction.

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